SUMMARY
This paper presents research on a non-symmetric damage element developed initially for the Ularc Finite Element Method (FEM) program. The paper outlines the mathematical development of the beam element, provides details of the use of the damage model in the FEM implementation, and sets out the limitations of the method. Recent mathematical research has focussed on implementing a symmetric finite element as part of the ULARC FEM package, which provides a simple but robust simple to test the mathematical implementation of damage theory on simple masonry structures with known failure methods.

INTRODUCTION
This paper presents the development of a beam - finite element for a singularly non-symmetric masonry block, based on the theory of damage mechanics. The FEM beam element predicts the peak strains for an initially elastic structure and then damaged structure. The theory of damage mechanics postulates a relationship between an increasing strain and the effective stiffness for a brittle material. The damage mechanics theory relates a scalar damage parameter, \( D \), to the change in the effective stiffness using equation (1).

\[
D = \frac{E - \bar{E}}{E}
\]  

(1)

The damage parameter has a domain from zero, representing an elastic material, to a value of one that represents a fully cracked material. The effective stiffness is defined as \( \bar{E} \) and the invariant elastic modulus as \( E \). Measurements have been made of the damage parameter for various materials (Krajcinovic 1996).

Nichols (2000) experimentally measured the change in the effective stiffness properties of pressed clay masonry panels that were subjected to a slowly increasing strain field. The strain field applied to the masonry panels was a non-proportional, constant, bi-axial, compression, and a displacement controlled harmonic shear strain. A linear relationship was derived between the increasing strain in the clay masonry panels and the experimentally measured changes in the effective stiffness (Nichols, 2000). The component of the strain field used as the dependent variable in the statistical analysis was the first principal strain, \( \varepsilon \), and the independent variable was the damage parameter, \( D \). The linear regression equation for estimating the damage parameter for pressed clay masonry panels is shown in equation (2).
Equation (2) provides a typical controlling convergence function. This function permits the implementation of a numerical algorithm to predict the stress and strains results for the damage accumulation in a theoretical structure modelled from a brittle material such as masonry. The FE algorithm is implemented within the computer program ULARC developed by Powell at the University of California - Berkeley in the 1970s (Sudhakar et al. 1972)).

A brief literature review outlines the development of previous damage mechanics based finite elements; the investigation into the damage parameterization for pressed masonry panels, the section on stiffness parameterization theory illustrates the use of standard matrix and integral methods to solve this problem and presents the stiffness co-efficients. The results section contains a summary of the relevant results equations. Addison (2004) clearly illustrates the uses of this technique in the analysis of retrofit alternatives of 16th century Scottish Church bridges.

LITERATURE REVIEW

Addison (2004) epitomizes in his recent talk the ethical dilemma facing all conservation engineers. The simple ethics question Addison posed related to saving a historic structure by introducing modern elements to the structure. Historic structures are not mausoleums. Historic structures are usually still part of a living community. Historic structures have been redeveloped or rebuilt since the start of recorded history. The conservation engineer generally needs simple analytical tools incorporating modern methods of analysis, which account for the properties of historic materials.

The last four decades has witnessed significant research in the use of numerical methods to study the inelastic behaviour of structures. One common inelastic method is the elastic-plastic finite element analysis technique, implemented in programs such as ULARC (Sudhakar et al. 1972) and also in the commonly available commercial FE packages. An alternative technique, still using Finite Element Method (FEM), is the use of the damage parameter theory to analyse deformations in structures made from a brittle material. The conceptual leap for damage mechanics is a monotonically increasing level of degradation of the stiffness of the element. The Boundary Element Method (BEM) provides for the static analysis of three-dimensional damaged structures with a degrading stiffness in the element. A brittle wall element (200 x 200 x 50 mm) was used to demonstrate the application of this BEM against published experimental results (Hatzigeorgiou and Beskos 2002). FEM applications, such as ULARC, provide a program capable of modelling degrading brittle materials.

ULARC permits direct input of the co-efficients for the flexibility matrix for a beam element. This program utilizes complex control structures allowing analysis techniques such as damage mechanics methods by avoiding the problems of singular matrices. The inversion routine developed by Sudhakar includes a sophisticated technique for ensuring that the stiffness matrix is not singular. This computer algorithm has not been implemented in other currently available commercial programs, and is one of the many features that make ULARC attractive as a code development platform (Sudhakar et al. 1972). The control structure implemented within ULARC can be modified to provide a convergent solution for the static analysis of brittle structures. The control structure uses equation (2) as the test function to determine
convergence of the solution. The form of one stress–strain function for a brittle material is shown in Figure 1.

![Figure 1 – Typical Stress Strain Curves for Brittle Materials](image)

This functional form for the brittle stress–strain curve was first sketched by Krajcinovic (1996) in his text on damage mechanics. Hatzigeorgiou and Beskos (2002) extended Krajcinovic’s model to include two alternative unloading paths that have residual strain. The gradients shown in Figure 1 are labelled E, 1, 2, and 3. The gradient E represents the initial stiffness or elastic modulus for the material. The theory of damage mechanics assumes that the elastic modulus is an invariant intrinsic property of the material. The gradient 1 represents an elastic unloading path, the gradient 2 represents an unloading path for a damage parameter model, and the gradient 3 represents a plastic unloading path.

The brittle curve of Figure 1 coupled with unloading path 2 illustrates the typical stress-strain curve observed in pure shear failure on a set of experimental masonry shear walls (Nichols 2000).

Hatzigeorgiou and Beskos used a damage parameter model for a BEM analysis that is equivalent to the model developed by Page (1979) in his study of bi-axial loaded masonry panel failure. The form of the brittle stress-strain loading curve depends on the test method. A tensile test method that uses force control will show catastrophic failure of the specimen at the peak applied load. A compressive test method will show a post peak drop in the stress with strain. This compressive feature results from interlock of the failing brittle pieces of the original material. The brittle loading curve shown in Figure 1 approximately represents a parabolic functional change in the effective stiffness typically based on a compressive test result.

No real world material behaves with an elastic unloading path when subjected to extreme loads such as great earthquakes. Krajcinovic showed, from experiments on Salem limestone, the strain limit on elastic unloading is about 25 µm for the first principal strain. Nichols (2000) experimentally investigated the damage parameterization curves for eight shear walls subjected to dynamic loads. Each clay masonry panel was 1.2 metres per side constructed with 110 mm thick bricks and a 1:1:6 cement-lime-sand mix by volume. The square, single
wythe masonry shear walls were damaged using a test rig capable of applying non-proportional constant bi-axial compression and harmonic shear. The shear was applied with a slowly increasing strain field. The dependent variable in the experimental analysis was the first principal strain assumed coincident with the first principal stress. Figure 2 shows the damage parameter results for the masonry panels with the strain as the dependent variable and the damage parameter as the independent variable. It also shows a least squares linear regression fit to the data points.

![damage results graph](image)

Figure 2 – Damage Results for the Eight Masonry Panels

Meek (1971) provides a standard discourse on the use of non-symmetric units within a general discussion on the analysis of beam units using matrix methods. As is quite common with these types of standard texts the examples use simple integral models that rarely reflect the geometrical forms of structural elements used in historic or modern buildings. Symbolic algebra techniques developed in the last 25 years provide a tool to rapidly and accurately solve the difficult integral equations generated in the analysis of non-symmetric beam elements (Abelson et al. 1987; Wolfram 2002). These techniques were applied to the algebra of symmetric beams incorporating a simple brittle damage model (Nichols 2002).

**STIFFNESS PARAMETERIZATION – MODEL DETAILS**

The simplest conceptual bending entity in a FE model is the beam element (Figure 3). The beam element is assumed typically to have an invariant Young's modulus, \(E\). The standard beam element for a typical structural analysis package has a constant moment of inertia, \(I\), end moments matrix, \(\mathbf{M}\), flexural stiffness matrix, \(\mathbf{K}\), and end rotations matrix, \(\mathbf{\Theta}\). The matrix equation relating the end moments to the end rotations for the prismatic beam element shown in Figure 3 is presented in equation (3).

\[
\mathbf{M} = \frac{EI}{L} \mathbf{K\Theta}
\]  

(3)
which can be presented in 2 x 2 array notation as shown in equation (4).

\[
\begin{bmatrix}
M_i \\
M_j
\end{bmatrix} = \frac{EI}{L} \begin{bmatrix}
K_{ii} & K_{ij} \\
K_{ji} & K_{jj}
\end{bmatrix} \begin{bmatrix}
\theta_i \\
\theta_j
\end{bmatrix}
\]  

(4)

The stone block of linearly varying depth or masonry buttress with the same shape property is a difficult integration task, with significantly greater mathematical complexity than is presented in standard structural analysis texts (Meek 1971). The integral bending equation for this geometry of stone block with a variable moment of inertia is shown in equation (5).

\[
\theta = \frac{1}{L} \int_0^L \frac{M}{EI(x)} x dx
\]  

(5)

The standard form of the inverse problem, for constant moment of inertia, to equation (4) is shown in equation (6).

\[
\begin{bmatrix}
I_{11} & I_{12} \\
I_{21} & I_{22}
\end{bmatrix} = \int_0^L \begin{bmatrix}
1 - \frac{x}{L} \\
\frac{x}{L}
\end{bmatrix} \frac{1}{EI} \begin{bmatrix}
1 - \frac{x}{L} & \frac{x}{L}
\end{bmatrix} dx
\]  

(6)

This problem was solved in 1980 (Figure 4) for standard steel I beam knee joint connections assuming the area of the flanges remained constant and the depth increased linearly to solve for knee joints in portal frames (Nichols 1981). This analysis provided an alternative to the assumption of a compression strut acting as a knee joint brace. This problem was solved in 2002 for a rectangular shaped masonry element subjected to a symmetric damage function (Nichols 2002).
The moment of inertia of the stone or masonry block varies along the member with the function presented in equation (7) and the relationship between the depth of section for the variable x is presented in equation (8).

\[ \psi = \frac{D_i}{D_j} - 1 \]  \hspace{1cm} (7)

\[ D_n = D_i(1-\eta) + \eta D_j \]  \hspace{1cm} (8)

Where \( I(\eta) \) is the function that encapsulated the ratio of the moment of inertia at the point \( \eta = \frac{x}{L} \). Substitution of (7) and (8) into equation (5) yields the following integral equation (9) for determining the variation in the moment of inertia along the beam. This solution was confirmed numerically using an Excel spreadsheet. The equation (9) is significantly more complex than the traditional assumption

\[ I(\eta) = \psi^3 \int_0^1 \eta^3 \ d\eta + 3\psi^2 \int_0^1 \eta^2 \ d\eta + 3\psi \int_0^1 \eta \ d\eta + \int_1^1 d\eta \]  \hspace{1cm} (9)

This equation (9) provides the relationship between the relative position in the beam and the moment of inertia. The relationship is considerably different to the standard relationships provided in the texts, which are usually an inverse relationship of the form shown in equation (8). The general slope equations established from the integrals presented in equation (5) are shown in equation (10).

\[ \phi_i = \frac{L}{EI} \int_0^1 \frac{M(\eta)}{I(\eta)} (1-\eta) \ d\eta \]  \hspace{1cm}  \text{and} \hspace{1cm} \phi_j = \frac{L}{EI} \int_0^1 \frac{M(\eta)}{I(\eta)} \eta \ d\eta \]  \hspace{1cm} (10)

The solution of the matrix of equations generated from equation (10) requires solution of an equation of the form shown in equation (11).
\[ \int_{0}^{1} \frac{f(\eta)}{a\eta^3 + b\eta^2 + c\eta + 1} d\eta \quad (11) \]

Where the function \( f(\eta) = n^2, (1-n)n, \) or \((1-n)^2\). These three integrals can be solved using an explicit integration technique because of the invertability requirement placed on the denominator of the integral in equation (11) due to the inertial mass of the object. A solution was obtained for a typical range of moment of inertia ratios to provide a graphical form for the results, using the symbolic notation program Mathematica (Wolfram 2002). The results are based on a moment equation shown in equation (12) based on the fixed end moments for the section in Figure 3.

\[ M(n) = (M_n) - (M_f(1-n)) \quad (12) \]

Two mathematical systems have been developed to cater for a degrading stiffness element with a known damage parameter model (Nichols 2002) and a section with varying moment of inertia. The purpose of the damage parameter model with varying stiffness extends beyond the simply study of masonry to include examples such as steel and stone elements (Krajcinovic, 1996). The ultimate objective is a generalized beam element model that has a varying moment of inertia, and a damage model that can cater for damage at any section of the element. The development of the generalized model will follow from this work and the development of a numerically or explicitly solvable form of equation (11) shown in equation (13).

\[ \int_{0}^{1} \frac{f(\eta)g(\eta)}{a\eta^3 + b\eta^2 + c\eta + 1} d\eta \quad (13) \]

Where \( g(\eta) \) is the inverse of the polynomial used to encapsulate the damage function (Nichols 2002) assuming that the damage in linearly independent of the moment of inertia. The current constraints imposed using the function \( g(\eta) \) are that all damage occurs within a single element, that the maximum element damage parameter, \( \lambda(0.5L) \), occurs at the midpoint of the element, and that the effective stiffness at each end remains equal to the Young’s modulus, \( \lambda(0) = E, \lambda(L) = E \). The damage function, \( \lambda(x) \), matches the constraints being smooth, differentiable, scalable, and symmetrical using a polynomial equation of sixth order as shown in equation (14). A set of polynomials and their inverse polynomial functions representing a damage level, \( D \), of 0.5 and 0.25 are shown in Figure 4.

\[ \lambda(x) = 1 - \sum_{i=0}^{6} a_i x^i \quad (14) \]

The next section of mathematics derives the variable stiffness matrix entries for use in the ULARC code for a damage element. Equation (1) cast as equation (15).

\[ g(x)E(x) = E \quad (15) \]

Equation (15) defines \( g(x) \) as the inverse of Kachanov’s integrity parameter (Krajcinovic, 1996). \( g(x) \) can be defined in polynomial form as listed in equation (16).
$$\bar{g}(x) = \sum_{i=0}^{6} b_i x^i$$

Equation (9) is obtained from equation (15) with the substitution of equation (16) for $\bar{g}(x)$ and with the substitution of equation (14) for $E(x)/E$ which yields equation (17).

$$\sum_{i=0}^{6} b_i x^i \sum_{i=0}^{6} a_i x^i = 1$$

The substitution of $x = nL$ into equation (16) derives the non-zero flexibility matrix entries from equation (6) which are shown in equations (18), (19), and (20). All non-zero matrix entries can be determined from these equations. The scalar function $n$ has a domain from 0 to 1, and the function $\bar{g}(x)$ transforms to $g(\eta)$ over the domain of $n$.

$$I_{11} = \frac{1}{a} \int_0^1 \frac{(1-\eta)^2 g(\eta)}{a \eta^4 + b \eta^2 + c \eta + 1} d\eta$$

$$I_{12} = \frac{1}{a} \int_0^1 \frac{((1-\eta)\eta)g(\eta)}{a \eta^4 + b \eta^2 + c \eta + 1} d\eta$$

$$I_{22} = \frac{1}{a} \int_0^1 \frac{(\eta^2)g(\eta)}{a \eta^4 + b \eta^2 + c \eta + 1} d\eta$$

Figure 4 – Varying Damage Modelled along a Beam Element with the Function, $\lambda(x)$.
$a = b = c = 0$ then with substitution into these two integral equations inversion of matrix $I$ yields the standard stiffness matrix entries of 4 and 2 for the matrix, $K$. The matrix entries for $K$ and $I$ for a variation in the damage parameter have been detailed elsewhere for the beam element with the constraint of peak damage at the center of the element (Nichols 2002).

**RESULTS**

The stiffness coefficients for the variable moment of inertia section shown in Figure 3 are presented in Figure 5.

![Figure 5 Stiffness Co-efficients variation with Moment of Inertia](image)

The equations for the variation in $K_{12}$ and $K_{11}$ are first order polynomials being $K_{12} = 3.7159I(\rho)-1.812$ and $K_{11} = 2.8897I(\rho)+1.118$ with $R^2$ of 0.99 for both equations. The best fit for an equation for the variation in $K_{22}$ is a power equation of the form $K_{22} = 3.99(I(\rho)^{2.2797})$ with $R^2$ of unity.

**CONCLUSIONS**

Analysis of historic masonry structures is an interesting challenge to provide an estimate of the capacity of the existing structure and then the capacity of any potential changes to the fabric of the structure. The critical element in any repair is matching the properties of the old and new materials. The challenge for the conservation engineer is in the adaptation of modern analysis techniques to old structures that are oft centuries old. This paper has investigated the application of the theory of damage mechanics to finite elements used in the analysis of beams by extending the theory to beam elements with a variable cross-section.

The cross section investigated for this research was a linear variation in the depth of section from the thinner to thicker end. An experienced masonry designer can use this type of element for the analysis of arch and other simple masonry structures. A linear variation in the $K_{12}$ and
$K_{11}$ elements were established for the section type and a power relationship for the matrix element $K_{22}$. The $K$ matrix entries are reduced to the standard numbers of four and two for a constant moment of inertia. Analysis was completed for the domain of the inertia ratio from one to two. A generalized equation for determination of the $K$ matrix entries has been provided for an element that has a variation in the moment of inertia and the effective stiffness with applied strain. This equation (number 13) permits the development of code for programs such as ULARC to model generalized masonry units provided that the functions defining the damage parameter variation and the moment of inertia are expressed as polynomials.

**ACKNOWLEDGEMENTS**

To Peter Kleeman, who never said this was easy.

**REFERENCES**


Nichols, J. M., Stiffness Co-efficients for Tapered Universal Beam Elements, *Original Figure - Sinclair Knight and Partners (16/6/1981)*, 1981.


