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where sum on repeated indices is implied but the limit corresponds to a fixed \( j \). The expression

\[
U_0 = \int_0^{\varepsilon_{ij}} \sigma_{ij} \, d\varepsilon_{ij}
\]

is the strain energy per unit volume or simply the strain energy density. The complementary strain energy density, \( U_0^* \), can be computed from

\[
U_0^* = \int_0^{\sigma_{ij}} \varepsilon_{ij} \, d\sigma_{ij}.
\]

The total internal work done by internal forces is given by the integral of the strain energy over the volume of the body, and it is denoted by \( U \):

\[
U = \int_V U_0 \, dV,
\]

and it represents the mechanical energy stored in the body. It is called the strain work of the body. The complementary strain energy \( U^* \) of an elastic body is defined as

\[
U^* = \int_V U_0^* \, dV.
\]

Analogous to Eq. (4.13), the strains may be derived from the complementary energy density function

\[
\varepsilon_{ij} = \frac{\partial U_0^*}{\partial \sigma_{ij}}.
\]

**Example 4.2** Consider the pin-connected structure shown in Fig. 4.6. The members of the truss are made of an elastic material whose uniaxial stress–strain behavior under tension and compression are given by

\[
\sigma = \begin{cases} 
E \sqrt{\varepsilon}, & \varepsilon \geq 0, \\
-E \sqrt{-\varepsilon}, & \varepsilon \leq 0.
\end{cases}
\]

\[E = \text{Young's modulus} \]
\[A_i = \text{cross-sectional area of the } i \text{th member}\]

![Figure 4.6 Pin-connected structure (truss).](image)
4.25 Write the virtual strain energy expression for the Euler–Bernoulli beam theory with the following nonlinear strain–displacement relation:

\[ \varepsilon_{xx} = \frac{du_0}{dx} + \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 - \frac{z}{dx^2}. \]

Express your answer in terms of the stress resultants. Do not assume.

4.26 The Timoshenko beam theory is based on the displacement field

\begin{align*}
  u(x, z) &= z\phi(x), \\
  w(x, z) &= w_0(x),
\end{align*}

where \( w_0 \) is the transverse deflection of a point on the midplane (i.e., \( z = 0 \)) of the plate and \( \phi \) is the rotation of a transverse normal line about the \( y \)-axis.

(i) Compute linear strains using the displacement field (a).

(ii) Write the expression for the total virtual work done by actual forces in moving through the virtual displacements \((\delta w_0, \delta \phi)\), assuming that the beam is loaded with a distributed transverse force \( q(x) \). Express the internal virtual work in terms of the stress resultants

\[ M = \int_A z\sigma_{xx} \, dA, \quad V = \int_A \sigma_{xz} \, dA. \]
we have

\[ F_i = \frac{\partial W_I}{\partial q_i} = \int_V \frac{\partial U_0}{\partial q_i} \, dV, \]

where \( F_i \) is a generalized force causing the generalized displacement \( q_i \). The generalized coordinate \( q_i \) can be a displacement or a rotation, implying that \( F_i \) can be a force or a moment. For linear elastic beams under axial and bending deformation, we can write

\[ F_i = \int_0^L \left[ \frac{E A}{dx} \frac{\partial}{\partial q_i} \left( \frac{du_0}{dx} \right) + E I \frac{d^2 w_0}{dx^2} \frac{\partial}{\partial q_i} \left( \frac{d^2 w_0}{dx^2} \right) \right] \, dx, \tag{5.1} \]

where it is understood that the axial displacement \( u_0 \) and transverse displacement \( w_0 \) can be expressed in terms of the generalized coordinates \( q_i \).

Two examples of application of the unit-dummy-displacement method are presented next.

**Example 5.2** Consider the truss shown in Fig. 5.1a. We wish to determine the vertical displacement \( v \) and horizontal displacement \( u \) of point \( O \) using the unit-dummy-displacement method. We shall assume linear elastic behavior and small strains.

Assume virtual (or dummy) displacements of \( \delta v_0 \) downward and \( \delta u_0 \) horizontal (see Fig. 5.1b) at point \( O \) so that \( \delta \mathbf{u}_0 = \delta u_0 \hat{e}_x + \delta v_0 \hat{e}_y \). The load in this case is \( \mathbf{R} = 0 \hat{e}_x + P \hat{e}_y \). Then we have

\[
\mathbf{R}_0 \cdot \delta \mathbf{u}_0 = 0 \cdot \delta u_0 + P \cdot \delta v_0 = \int_V \sigma_{ij} \delta \varepsilon_{ij} \, dV = \int_0^{h_1} A^{(1)} \sigma_{11}^{(1)} \delta \varepsilon_{11}^{(1)} \, dx + \int_0^{h_2} A^{(2)} \sigma_{11}^{(2)} \delta \varepsilon_{11}^{(2)} \, dx,
\]

where \( A^{(1)} = A^{(2)} = A, h_1 = h, \) and \( h_2 = \sqrt{2}h \). The actual stresses are computed in terms of the actual displacements \( u_0 \) and \( v_0 \) of point \( O \) from the deformed geometry.

![Figure 5.1](image-url) (a) Original truss. (b) Truss with virtual displacements.
Thus \( w_L \equiv c_2 \) and \( \theta_L \equiv c_3 \) can be used as the generalized coordinates. Solving the above equations for \( a_3 \) and \( a_4 \) in terms of \( w_L \) and \( \theta_L \), and substituting the result into Eq. (5.25), we obtain

\[
w_0(x) = \left(3 \frac{x^2}{L^2} - 2 \frac{x^3}{L^3}\right) c_2 + L \left(-\frac{x^2}{L^2} + \frac{x^3}{L^3}\right) c_3.
\]

Now using the unit-dummy-load method, we obtain

\[
F_0 = \int_0^L EI \frac{d^2 w_0}{dx^2} \frac{d}{dc_2} \left(\frac{d^2 w_0}{dx^2}\right) dx = \int_0^L EI \left[\left(\frac{6}{L^2} - \frac{12}{L^2} \frac{x}{L}\right) c_2 + L \left(-\frac{2}{L^2} + \frac{6}{L^2} \frac{x}{L}\right) c_3\right] \left(\frac{6}{L^2} - \frac{12}{L^2} \frac{x}{L}\right) dx,
\]

\[
M_0 = \int_0^L EI \frac{d^2 w_0}{dx^2} \frac{d}{dc_3} \left(\frac{d^2 w_0}{dx^2}\right) dx = \int_0^L EI \left[\left(\frac{6}{L^2} - \frac{12}{L^2} \frac{x}{L}\right) c_2 + L \left(-\frac{2}{L^2} + \frac{6}{L^2} \frac{x}{L}\right) c_3\right] L \left(-\frac{2}{L^2} + \frac{6}{L^2} \frac{x}{L}\right) dx.
\]

Upon carrying out the integration, we obtain

\[
F_0 = \frac{12EI}{L^3} c_2 - \frac{6EI}{L^2} c_3, \quad M_0 = -\frac{6EI}{L^2} c_2 + \frac{4EI}{L} c_3,
\]

whose solution gives the deflection and rotation of the free end:

\[
c_2 = w_L = w_0(L) = \frac{F_0 L^3}{3EI} + \frac{M_0 L^2}{2EI}, \quad c_3 = \theta_L = \left(\frac{dw_0}{dx}\right)_{x=L} = \frac{F_0 L^2}{2EI} + \frac{M_0 L}{EI}.
\]

The procedure discussed in the above example to express the displacements \( u_0(x) \) and \( w_0(x) \) in terms of unknown generalized displacement degrees of freedom holds for any bar and beam structure. When a distributed load \( q(x) \neq 0 \), one may convert it to a set of statically equivalent point loads \( F_i \) acting at the same points and directions as the generalized displacements \( c_i \) of the beam. For bars and beams with constant \( EA \) and \( EI \) but arbitrary load \( q(x) \), this procedure results in exact values of the generalized displacements \( c_i \). The procedure can be generalized to bars and beams with arbitrary boundary conditions, geometric properties, and material properties. However, in such cases, the solutions obtained are approximate.

To further explain the idea, we express the displacement in terms of a linear combination of undetermined parameters \( c_i \) and known functions \( \varphi_i \) that satisfy the kinematic boundary conditions:

\[
w_0(x) \approx \sum_{i=1}^N c_i \varphi_i(x). \tag{5.25}
\]
or

\[
\begin{bmatrix}
\frac{12EI}{L^3} + k & \frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\begin{bmatrix}
u_3 \\
u_4
\end{bmatrix}
= \frac{q_0L}{12}
\begin{bmatrix}
6 \\
L
\end{bmatrix}.
\]

Solving for \( u_3 \) and \( u_4 \) by Cramer’s rule, we obtain

\[
u_3 = w_0(L) = \frac{q_0L^4}{8EI} \frac{1}{(1 + (kL^3/3EI))},
\]

\[
u_4 = -\frac{q_0L^3}{6EI} \frac{(EI - (kL^3/24))}{(EI + (kL^3/3))}.
\]

Note that when \( k = 0 \), we obtain the deflection \( u_3 \) and rotation \( u_4 \) at the free end of a cantilever beam under uniformly distributed load of intensity \( q_0 \).

5.3 PRINCIPLES OF VIRTUAL FORCES AND COMPLEMENTARY POTENTIAL ENERGY

5.3.1 Principle of Virtual Forces

In Section 5.1 we discussed the principle of virtual displacements. Naturally, the virtual work done by virtual forces in moving through actual displacements should have similar use. Here we formulate the principle of virtual forces. The basic idea is that each particle in the body undergoes actual displacements until the stresses developed satisfy the equations of equilibrium.

Consider single-valued, differentiable variations of a stress field \( \delta \sigma \) and body forces \( \delta f \) that satisfy the linear equilibrium equations both within the body and on its boundaries:

\[
\nabla \cdot \delta \sigma + \delta f = 0 \quad \text{in } V,
\]

\[
\hat{n} \cdot \delta \sigma = \delta t \quad \text{on } S_2.
\]

(5.51)

(5.52)

We shall call such a stress field a \textit{statically admissible field of variation}. These virtual stresses and forces, except for \textit{self-equilibrating}, are completely arbitrary and independent of the true stresses and forces.

The external \textit{complementary virtual work} is defined by

\[
\delta W^*_E = - \int_V \mathbf{u} \cdot \delta \mathbf{f} \, dV - \int_{S_1} \mathbf{u} \cdot \delta \mathbf{t} \, dS,
\]

(5.53a)

where \( \mathbf{u} \) is the displacement vector, and \( \delta \mathbf{f} \) and \( \delta \mathbf{t} \) satisfy Eqs. (5.51) and (5.52). The internal \textit{complementary virtual work} is given by

\[
\delta W^*_I = \int_V \varepsilon \cdot \delta \sigma \, dV,
\]

(5.53b)
where \( \varepsilon \) are actual linear strains and \( \delta \sigma \) is the virtual stress field that satisfies Eqs. (5.51) and (5.52).

The principle of complementary virtual work (or virtual forces) states that the strains and displacements in a deformable body are compatible and consistent with the constraints if and only if the total complementary virtual work is zero:

\[
\delta W_I^* + \delta W_E^* = 0. \tag{5.54}
\]

It can be shown that the principle of virtual forces gives the kinematic relations and geometric boundary conditions as the Euler equations. This is illustrated by considering the rectangular component form of \( \delta W_I^* \) and \( \delta W_E^* \) for a deformable solid:

\[
0 = \int_V \varepsilon_{ij} \delta \sigma_{ij} 
\]

\[
-\int_V \varepsilon_{ij} \delta \sigma_{ij} 
\]

\[
\delta \sigma_{ij} dV + \int_{S_1} (u_i - \hat{u}_i) n_j \delta \sigma_{ij} dS,
\]

where sum on repeated indices is assumed. Because \( \delta \sigma_{ij} \) is arbitrary, we obtain the strain–displacement equations and the displacement boundary conditions as the Euler equations

\[
\varepsilon_{ij} - \frac{1}{2} (u_{i,j} + u_{j,i}) = 0 \quad \text{in } V \tag{5.56a}
\]

\[
u_i - \hat{u}_i = 0 \quad \text{on } S_1. \tag{5.56b}
\]

### 5.3.2 Unit-Dummy-Load Method

The unit-dummy-load method is a special case of the complementary virtual work, and it can be used to determine displacements and forces in structures. The basic idea can be described by analogy with the unit-dummy-displacement method. If \( u_0 \) is the true displacement at point \( O \) in an elastic structure, we can prescribe a virtual force \( \delta R_0 \) at the point. The application of virtual force induces a system of virtual stresses \( \delta \sigma_{ij} \) that satisfy the equilibrium equations. Then from the principle of virtual forces (5.54), we have \( \delta W_E = -u_0 \delta R_0 \):

\[
u_0 \delta R_0 = \int_V \varepsilon_{ij} \delta \sigma_{ij}^0 dV. \tag{5.57}
\]

Once again, one can take \( \delta R_0 = 1 \) and calculate corresponding virtual internal stresses \( \delta \sigma_{ij}^0 \). Equation (5.57) represents the unit-dummy-load method.

For the Euler–Bernoulli beams, Eq. (5.57) takes the form

\[
u_0 \delta P_0 + w_0 \delta F_0 + \theta_0 \delta M_0 + \phi_0 \delta T_0
\]

\[
l = \int_0^L \left( \frac{N}{EA} \delta N + \frac{M}{EI} \delta M + f_s \frac{V}{GA} \delta V + \frac{T}{GJ} \delta T \right) dx,
\]

where \( \varepsilon \) are actual linear strains and \( \delta \sigma \) is the virtual stress field that satisfies Eqs. (5.51) and (5.52).
5.4 PRINCIPLE OF COMPLEMENTARY POTENTIAL ENERGY AND CASTIGLIANO’S THEOREM II

Noting that the complementary potential energy due to virtual loads, and the complementary strain energy, are given by

$$\delta V^* = \delta W_E^*, \quad \delta U^* = \delta W_I^*,$$

we can arrive at the principle of maximum complementary potential energy:

$$\delta \Pi^* \equiv \delta(U^* + V^*) = 0,$$

where $\Pi^*$ denotes the total complementary potential energy.

Analogous to the unit-dummy-load method, we can derive Castigliano’s Theorem II from the principle of maximum complementary energy. We have

$$\delta \Pi^* \equiv \delta U^* + \delta V^* = 0 \rightarrow \delta U^* = -\delta V^*.$$  

If $U^*$ and $V^*$ can be expressed in terms of point loads $F_i$, then we have

$$\delta U^* = \frac{\partial U^*}{\partial F_i} \delta F_i, \quad \delta V^* = \frac{\partial V^*}{\partial F_i} \delta F_i = -u_i \delta F_i,$$

and

$$\left(\frac{\partial U^*}{\partial F_i} - u_i\right) \delta F_i = 0 \quad \text{or} \quad \frac{\partial U^*}{\partial F_i} = u_i.$$  

Equation (5.67) represents Castigliano’s Theorem II. Equation (5.67) is valid for structures that are linearly elastic as well as nonlinearly elastic. When the material of the structure is linearly elastic, we have $U_0 = U_0^*$ and $U = U^*$ in value. However, $U$ is always expressed in terms of displacements while $U^*$ is in terms of forces, and Castigliano’s Theorem I is based on $U$ while Theorem II is based on $U^*$.

Castigliano’s Theorem II, which is essentially the same as the unit-dummy-load method, is used to determine deflections and slopes under applied point forces and moments. However, when a structure does not have a load (or moment) at the point at which displacement (or slope) is required, the determination of the displacement (or slope) at that point requires the use of a fictitious (or dummy) load. For example, consider a beam that is subjected to uniformly distributed load $q_0$, and point loads $F_1, F_2, \ldots$, etc. Suppose that we wish to determine the vertical deflection $w_0$ at a point at which there is no point load. We introduce a fictitious vertical load $R$ at the point, and then write the complementary strain energy in terms of $q_0, R,$ and $F_1, F_2, \ldots$, etc. Then we use Castigliano’s Theorem II to determine the desired displacement at the point:

$$w_0 = \left. \frac{\partial U^*}{\partial R} \right|_{R=0}.$$
and the only nonzero strain is (nonlinear)
\[
\varepsilon_{xx}(x, y, z) = \frac{du_0}{dx} + \frac{1}{2} \left( \frac{dw_0}{dx} \right)^2 - z \frac{d^2w_0}{dx^2} - y \frac{d^2w_0}{dx^2}.
\]

Here \(u_0\) and \(w_0\) are the axial and transverse displacements of a point \((x, y, 0)\) in the beam. Your answer should be expressed in terms of the area-integrated quantities [e.g., stress resultants of Eq. (5.9)].

5.4 The Timoshenko beam theory. The Euler–Bernoulli beam theory is based on the assumption that a straight line transverse to the axis of the beam before deformation remains (i) straight, (ii) inextensible, and (iii) normal to the midplane after deformation. In the Timoshenko beam theory, the first two assumptions are kept but the normality condition is relaxed by assuming that the rotation is independent of the slope \((dw_0/dx)\) of the beam. Using these assumptions, the displacement field of the beam can be expressed as

\[
\begin{align*}
  u(x, z) &= u_0(x) + z\phi(x) \\
  v &= 0 \\
  w &= w_0(x),
\end{align*}
\]

where \((u, v, w)\) are the displacements of a point along the \((x, y, z)\) coordinates, \((u_0, w_0)\) are the displacements of a point on the midplane of an undeformed beam, and \(\phi\) is the rotation (about the \(y\)-axis) of a transverse normal line. Derive the equations of equilibrium of the Timoshenko beam theory using the principle of virtual displacements for the case of infinitesimal strains. Also derive the natural boundary conditions of the theory.

5.5 Use the unit-dummy-displacement method to determine the displacement in the direction of the applied load \(P\) in the structure in Exercise 4.10.

5.6 Use Castigliano’s Theorem I to determine the axial forces in the wires of the structure shown in Fig. E5.6. Assume small deformation.

5.7 Use Castigliano’s Theorem I to determine the unknown generalized displacements and forces of the beam shown in Fig. E5.7. Use the results of Example 5.7,6.
the present case is given by (r = \text{constant})

\[ \nabla \cdot \mathbf{v} = 0 \]

Thus the kinetic and potential energy are

\[ K = \frac{m}{2}(l^2 \dot{\theta}^2) \]

\[ \delta K = ml^2 \dot{\theta} \delta \theta \]

Therefore, the Lagrangian function equation is given by

\[ \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta} \]

which yields

\[ -mg l \sin \theta - \frac{d}{dt}(ml^2 \dot{\theta}) = 0 \]

Equation (6.19) represents a second order differential equation for \( \theta \). For small angular motions, Equation (6.19) is

Now suppose that the mass experiences a linear velocity (i.e., the pendulum is suspended in a flowing fluid). According to Stoke’s law,

\[ \mathbf{F} = -\mu \mathbf{v} \]

where \( \mu \) is the viscosity of the surrounding fluid, \( \mathbf{v} \) is the linear velocity of the massless rod supporting the pendulum bob, and \( \mathbf{v} \) is the unit vector in the direction of the mass. \( \mathbf{v} \) and \( \mathbf{v} \) are orthogonal. The kinetic energy \( K \) of the pendulum is given from a potential function. Thus, the dynamical system is (split) into two parts, one (the kinetic force) conservative and the other (the damping force) non-conservative and the other (the damping force) non-conservative. Therefore, we use Hamilton’s principle expressed as

\[ \delta W_E = \delta V - \mathbf{F} \cdot (l \delta \dot{\theta}) \]

Then the equation of motion is given by

\[ -\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{\theta}} \right) + \mathbf{Q} = 0 \]

The coefficient \( c = 6\pi a\mu/m \) is
6.4 Derive the equations of motion for the double pendulum of Exercise 4.1.
6.5 Derive the equations of motion for the rigid-body assemblage of Exercise 4.2.
6.6 Consider a pendulum of mass $m_1$ with a flexible suspension, as shown in Fig. E6.6. The hinge of the pendulum is in a block of mass $m_2$, which can move up and down between the frictionless guides. The block is connected by a linear spring (of spring constant $k$) to an immovable support. The coordinate $x$ is measured from the position of the block in which the system remains stationary. Derive the Euler–Lagrange equations of motion for the system.

![Figure E6.6](image)

6.7 Consider a block of mass $m_2$ sliding on another block of mass $m_1$, which in turn slides on a horizontal surface, as shown in Fig. E6.7. Using $u_1$ and $u_2$ as coordinates, obtain the equations of motion. Assume that all surfaces are frictionless.

![Figure E6.7](image)

6.8 Figure E6.8 represents a double pendulum that is suspended from a block which moves horizontally with a prescribed motion, $x = f(t)$. Derive the Euler–Lagrange equations of motion of the pendulum when it oscillates in the $(x, y)$-plane under the action of gravity and prescribed motion.

6.9 Two masses $m_1$ and $m_2$ are attached to the ends of an inextensible cord that is suspended over a frictionless stationary pulley, as shown in Fig. E6.9. Find the equations of motion of the system.

6.10 Repeat Exercise 6.9 for the case in which a monkey of mass $m_3$ is climbing up the cord above mass $m_1$ with a speed $v_0$ relative to mass $m_1$ (see Fig. E6.10).

6.11 Determine the motion of all masses in the suspended double-pulley problem represented in Fig. E6.11.
that the total potential energy principle or weak form of the weak form approximation.

7.21 A cable suspended between points $A : (0, 0)$ and $B : (L, 0)$ subjected to uniformly distributed transverse load of intensity $q$. 

7.22 A cantilever beam subjected to uniformly distributed transverse load of intensity $q$. 

7.23 The symmetric half of the simply supported beam of Example 7.7 ($0 \leq x \leq L/2$). 

7.24 A beam clamped at the left end and simply supported at the right end subjected to point load $F_0$ at $x = L/2$. 

7.25 A simply supported beam with a spring support at $x = L/2$ subjected to uniformly distributed load of intensity $q_0$. 

7.26 A square elastic membrane fixed on all its sides and subjected to uniformly distributed load of intensity $f_0$. 

7.27 A quadrant model (because of the biaxial symmetry) of Exercise 7.26. 

7.28–7.34 Find the two-parameter Ritz approximation for each of the problems in Exercises 7.21–7.27 and compare the results with the exact solutions when possible. 

7.35 Find the first two natural frequencies of a cantilever beam with uniformly distributed load of intensity $b(x) = ax + bx^2$, where $a$ and $b$ are constants. 

7.36 Find a two-parameter Ritz approximation of the deflection of a simply supported beam on an elastic foundation subjected to uniformly distributed load. Use (a) algebraic polynomials. 

7.37 Derive the matrix equations corresponding to the Ritz approximation

$$W_N = c_1x^2 + c_2x^3 + \cdots + c_nx^n$$

of a cantilever beam with a uniformly distributed load $b_j$ in explicit form in terms of $i$, $j$, $L$, $EI$, and $q$. 

7.38 Use a two-parameter Ritz approximation with the assumption that $M = 0$ to determine the critical buckling load $P$ of a simple beam. 

7.39 Consider the buckling of a uniform beam according to the Euler-Bernoulli theory. The total potential energy functional for the system is

$$\Pi (w_0, \phi_x) = \frac{1}{2} \int_0^L \left[ D \left( \frac{d \phi_x}{dx} \right)^2 + S \left( \frac{dw_0}{dx} \right)^2 + N \frac{d \phi_x}{dx} \frac{dw_0}{dx} \right] dx$$

where $w_0(x)$ is the transverse deflection, $\phi_x$ is the axial displacement, $D$ is the bending stiffness, $S$ is the shear stiffness, and $N$ is the axial force, to determine the critical buckling load $N_{cr}$ of a simply supported beam.
using the following $N$-parameter Galerkin approximation:

$$U_N = \sum_{i,j=1}^{N} c_{ij} \sin i\pi x \sin j\pi y.$$ 

\[ \text{7.55} \] Solve the nonlinear equation in Exercise 7.52 by the Galerkin method.

\[ \text{7.56} \] Find a two-parameter Galerkin solution of a clamped (at both ends) beam under uniformly distributed load.

\[ \text{7.57} \] Solve the equation in Exercise 7.49 using the least-squares method.

\[ \text{7.58} \] Solve the problem of Exercise 7.52 using the least-squares method.

\[ \text{7.59} \] Solve the equation in Exercise 7.53 using the least-squares method.

\[ \text{7.60} \] Solve the equation in Exercise 7.54 using the least-squares method.

\[ \text{7.61} \] Consider a cantilever beam of variable flexural rigidity, $EI = a_0[2 - (x/L)^2]$, and carrying a distributed load, $q = q_0[1 - (x/L)]$. Find a three-parameter solution using the collocation method. Use $x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

\[ \text{7.62} \] Repeat Exercise 7.52 using the collocation method.

\[ \text{7.63} \] Solve the differential equation in Exercise 7.53 by the collocation method.

\[ \text{7.64} \] Solve the problem in Exercise 7.54 using the one-point collocation method.

\[ \text{7.65} \] Consider the Laplace equation

$$-\nabla^2 u = 0, \quad 0 < x < 1, \quad 0 < y < \infty,$$

$$u(0, y) = u(1, y) = 0 \quad \text{for} \quad y > 0,$$

$$u(x, 0) = x(1 - x), \quad u(x, \infty) = 0, \quad 0 \leq x \leq 1.$$

Assuming an approximation of the form

$$u(x, y) = c_1(y)x(1 - x),$$

find the differential equation for $c_1(y)$ and solve it exactly.

\[ \text{7.66} \] Use the semidiscretization method to find a two-parameter approximation of the form

$$u(x, y) = c_1(y)\cos \frac{\pi x}{2a} + c_2(y)\cos \frac{3\pi x}{2a},$$

to determine an approximate solution of the torsion problem in Example 7.16.

\[ \text{7.67} \] Use the semidiscretization method to find a two-parameter approximation of the form

$$w(x, t) = c_1(t)(1 - \cos 2\pi x),$$
the left-end node (i.e., node 1), and then use the interpolation property of $\psi_i$,

$$\psi_i(\bar{x}_j) = \delta_{ij}, \quad i, j = 1, 2, 3, 4,$$

to write $\psi_i$ as a polynomial that vanishes at all $\bar{x}_j, \bar{x}_j \neq \bar{x}_i$,

$$\psi_1(\bar{x}) = c_1(\bar{x} - \bar{x}_2)(\bar{x} - \bar{x}_3)(\bar{x} - \bar{x}_4),$$

$$\psi_2(\bar{x}) = c_2(\bar{x} - \bar{x}_1)(\bar{x} - \bar{x}_3)(\bar{x} - \bar{x}_4),$$

and determine $c_1, c_2$, etc., such that $\psi_1(\bar{x}_1) = 1, \quad \psi_2(\bar{x}_2) = 1$, etc.

Use the minimum possible number of linear finite elements to analyze the axially loaded structures of Exercises 9.3–9.9.

**9.3** Find the stresses and compressions in each section of the composite member shown in Fig. E9.3. Use $E_s = 30 \times 10^6$ psi, $E_b = 10^7$ psi, $E_b = 15 \times 10^6$ psi, and the minimum number of linear finite elements.

![Figure E9.3](image)

**9.4** A solid circular brass cylinder ($E_b = 15 \times 10^6$ psi, $d_b = 0.25$ in.) is encased in a hollow circular steel shell ($E_s = 30 \times 10^6$ psi, $d_s = 0.25$ in.). A load of $P = 1,330$ lb compresses the assembly as shown in Fig. E9.4. Determine (a) the compression and (b) the compressive forces and stresses in the steel shell and brass cylinder. Use the minimum number of linear finite elements. Assume that the Poisson effect is negligible.

**9.5** A rectangular steel bar ($E_s = 30 \times 10^6$ psi) of length 24 in. has a slot in the middle half of its length, as shown in Fig. E9.5. Determine the displacement of the ends due to the axial loads $P = 2,000$ lb. Use the minimum number of linear elements.

**9.6** The two members in Fig. E9.6 are fastened together and to rigid walls. If the members are stress-free before they are loaded, what will be the stresses and deformations in each after the two 50,000 lb. loads are applied? Use $E_s = 30 \times 10^6$ psi and $E_d = 10^7$ psi; the aluminum rod is 2 in. in diameter and the steel rod is 1.5 in. in diameter.
Interestingly, the stiffness matrix of the mixed finite element model with linear interpolation of both \( w_0 \) and \( M \) is the same as that of the displacement finite element model derived in Section 9.3 using the \( C^1 \) (Hermite cubic) interpolation. However, the load vector differs in the sense that the mixed model does not contain contributions of distributed load \( q(x) \) to the nodal moments.

**Weighted-Residual Finite Element Models** It is possible to construct a mixed finite element model of Eqs. (10.77a,b) using a weighted-residual formulation. As will be seen shortly, this model requires higher-order approximations of both \( w_0 \) and \( M \) because they must satisfy both essential and natural boundary conditions. This leads to a complicated set of finite element equations that are not practical. Here we present the main ideas behind the development of the model.

The weighted-residual statement of Eqs. (10.77a,b) is

\[
0 = \int_{x_a}^{x_b} v_1 \left( -\frac{d^2 w_0}{dx^2} - \frac{M}{EI} \right) \, dx,
\]

\[
0 = \int_{x_a}^{x_b} v_2 \left( -\frac{d^2 M}{dx^2} - q \right) \, dx,
\]

where \((v_1, v_2)\) are the weight functions. A close examination of the above statements indicate that \( v_1 \sim M \) and \( v_2 \sim w_0 \). Using approximations of the form

\[
w_0(x) \approx \sum_{i=1}^{4} \Delta_i \varphi_i^{(1)}(x), \quad M(x) \approx \sum_{i=1}^{4} \Lambda_i \varphi_i^{(2)}(x),
\]

we obtain the following Galerkin (i.e., \( v_1 \sim \varphi_i^{(2)} \) and \( v_2 \sim \varphi_i^{(1)} \)) finite element model:

\[
\begin{bmatrix}
0 \\
[B^e]
\end{bmatrix}
\begin{bmatrix}
[A^e] \\
[D^e]
\end{bmatrix}
\begin{bmatrix}
\{\Delta^e\} \\
\{\Lambda^e\}
\end{bmatrix}
= \begin{bmatrix}
\{f^e\} \\
\{0\}
\end{bmatrix},
\]

where

\[
A^e_{ij} = \int_{x_a}^{x_b} \varphi_i^{(1)} \frac{d^2 \varphi_j^{(2)}}{dx^2} \, dx,
\]

\[
f^e_i = -\int_{x_a}^{x_b} q \varphi_i^{(1)} \, dx,
\]

\[
B^e_{ij} = \int_{x_a}^{x_b} \varphi_i^{(2)} \frac{d^2 \varphi_j^{(1)}}{dx^2} \, dx,
\]

\[
D^e_{ij} = \int_{x_a}^{x_b} \varphi_i^{(2)} \frac{d^2 \varphi_j^{(2)}}{dx^2} \, dx.
\]

For Hermite cubic interpolation of both \( w_0 \) and \( M \), we have \( \varphi_i^{(1)} = \varphi_i^{(2)} \). The coefficient matrix in Eq. (10.91) is not symmetric.
The least-squares finite element model of the pair (10.77a,b) is based on the variational statement

\[
0 = \delta \int_{x_a}^{x_b} \left[ p_1 \left( \frac{d^2 w_0}{dx^2} + \frac{M}{EI} \right)^2 + p_2 \left( \frac{d^2 M}{dx^2} + q \right)^2 \right] dx
= 2 \int_{x_a}^{x_b} \left[ p_1 \left( \frac{d^2 w_0}{dx^2} + \frac{M}{EI} \right) \left( \frac{d^2 \delta w_0}{dx^2} + \frac{\delta M}{EI} \right) + p_2 \left( \frac{d^2 M}{dx^2} + q \right) \frac{d^2 \delta M}{dx^2} \right] dx,
\]

or

\[
0 = \int_{x_a}^{x_b} \frac{d^2 \delta w_0}{dx^2} \left( \frac{d^2 w_0}{dx^2} + \frac{M}{EI} \right) dx, \quad (10.93)
0 = \int_{x_a}^{x_b} \left[ \frac{\delta M}{EI} \left( \frac{d^2 w_0}{dx^2} + \frac{M}{EI} \right) + \frac{EIp_2}{p_1} \frac{d^2 \delta M}{dx^2} \left( \frac{d^2 M}{dx^2} + q \right) \right] dx, \quad (10.94)
\]

where \((p_1, p_2)\) are weights used to make the entire statement have the same units. Substituting the approximation (10.90) into Eqs. (10.93) and (10.94), we obtain

\[
\begin{bmatrix}
[A^e] & [B^e]^T
\end{bmatrix}
\begin{bmatrix}
\{\Delta^e\}
\end{bmatrix}
= \begin{bmatrix}
[q^e]
\end{bmatrix}
= \begin{bmatrix}
0
\end{bmatrix},
\]

where

\[
A_{ij}^e = \int_{x_a}^{x_b} \left( \frac{EIp_2}{p_1} \frac{d^2 \varphi_i^{(2)}}{dx^2} \frac{d^2 \varphi_j^{(2)}}{dx^2} + \left( \frac{\varphi_i^{(2)}}{\varphi_j^{(2)}} \right)^2 \right) dx,
q_i^e = - \int_{x_a}^{x_b} \frac{d^2 \varphi_i^{(2)}}{dx^2} q dx,
B_{ij}^e = \int_{x_a}^{x_b} \frac{\varphi_i^{(2)}}{\varphi_j^{(2)}} d^2 \varphi_j^{(1)} dx,
D_{ij}^e = \int_{x_a}^{x_b} \frac{d^2 \varphi_i^{(1)}}{dx^2} \frac{d^2 \varphi_j^{(1)}}{dx^2} dx. \quad (10.96)
\]

Obviously, \(p_1\) and \(p_2\) should be selected such that \(EI(p_2/p_1)\) has the dimension of \(1/L^2\).

We close this section with the comment that finite element models other than the displacement finite element model are not commonly used in practice. In fact, finite element models based on the Timoshenko beam theory that accounts for transverse shear strain \(\gamma_{xz}\) may be used to analyze beams irrespective of whether shear deformation is significant. The next section is devoted to the discussion of Timoshenko beam finite elements.
Finite Element Model  Let \((w_0, M_{xx}, M_{yy}, M_{xy})\) be interpolated by expressions of the form

\[
\begin{align*}
    w_0 &= \sum_{i=1}^{r} w_i \psi_i^{(1)}, \\
    M_{xx} &= \sum_{i=1}^{s} M_{xi} \psi_i^{(2)}, \\
    M_{yy} &= \sum_{i=1}^{p} M_{yi} \psi_i^{(3)}, \\
    M_{xy} &= \sum_{i=1}^{q} M_{xyi} \psi_i^{(4)},
\end{align*}
\]

(10.134)

where \(\psi_i^{(\alpha)}, (\alpha = 1, 2, 3, 4)\) are appropriate interpolation functions. Substituting Eq. (10.134) into Eqs. (10.128)–(10.131) [or substituting Eq. (10.134) into \(J_1\) and setting the partial variations of \(J_1\) with respect to \(w_j, M_{xj}, M_{yj}, \text{ and } M_{xyj}\) to zero separately], we obtain

\[
\begin{bmatrix}
    [K^{11}] & [K^{12}] & [K^{13}] & [K^{14}] \\
    [K^{22}] & [K^{23}] & [K^{24}] \\
    \text{symm.} & [K^{33}] & [K^{34}] & [K^{44}]
\end{bmatrix}
\begin{bmatrix}
    \{w\} \\
    \{M_{x}\} \\
    \{M_{y}\} \\
    \{M_{xy}\}
\end{bmatrix}
= 
\begin{bmatrix}
    \{F^{1}\} \\
    \{F^{2}\} \\
    \{F^{3}\} \\
    \{F^{4}\}
\end{bmatrix},
\]

(10.135)

where

\[
\begin{align*}
    K_{ij}^{11} &= 0, \quad i, j = 1, 2, \ldots, r, \\
    K_{ij}^{12} &= \int_{\Omega_e} \frac{\partial \psi_i^1}{\partial x} \frac{\partial \psi_j^2}{\partial x} \, dxdy, \quad i = 1, 2, \ldots, r; \quad j = 1, 2, \ldots, s, \\
    K_{ij}^{13} &= \int_{\Omega_e} \frac{\partial \psi_i^1}{\partial y} \frac{\partial \psi_j^3}{\partial y} \, dxdy, \quad i = 1, 2, \ldots, r; \quad j = 1, 2, \ldots, p, \\
    K_{ij}^{14} &= \int_{\Omega_e} \left( \frac{\partial \psi_i^1}{\partial x} \frac{\partial \psi_j^4}{\partial y} + \frac{\partial \psi_i^1}{\partial y} \frac{\partial \psi_j^4}{\partial x} \right) \, dxdy, \quad i = 1, 2, \ldots, r; \quad j = 1, 2, \ldots, q, \\
    K_{ij}^{22} &= \int_{\Omega_e} (-\bar{D}_{22}) \psi_i^2 \psi_j^2 \, dxdy, \quad i, j = 1, 2, \ldots, s, \\
    K_{ij}^{23} &= \int_{\Omega_e} (-\bar{D}_{12}) \psi_i^2 \psi_j^3 \, dxdy, \quad i = 1, 2, \ldots, s; \quad j = 1, 2, \ldots, p, \\
    K_{ij}^{24} &= 0, \quad i = 1, 2, \ldots, s; \quad j = 1, 2, \ldots, q, \\
    K_{ij}^{33} &= \int_{\Omega_e} (-\bar{D}_{31}) \psi_i^3 \psi_j^3 \, dxdy, \quad i, j = 1, 2, \ldots, p, \\
    K_{ij}^{34} &= 0, \quad i = 1, 2, \ldots, p; \quad j = 1, 2, \ldots, q, \\
    K_{ij}^{44} &= \int_{\Omega_e} (D_{66})^{-1} \psi_i^4 \psi_j^4 \, dxdy, \quad i, j = 1, 2, \ldots, q,
\end{align*}
\]

(10.136)
cylindrical coordinate system. Next, carry out the operations indicated in the equilibrium equation (i.e., take the divergence of the stress tensor, noting the dot product between the basis vectors) and collect the coefficients of various base vectors to obtain the required result.

3.33 The linear strain tensor in vector form is defined by

\[
\hat{\varepsilon} = \frac{1}{2} \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right]
\]

where

\[
\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{1}{r} \hat{\mathbf{e}}_\theta \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}, \quad \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \hat{\mathbf{e}}_\theta, \quad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r.
\]

Using Eq. (2) and the displacement vector

\[
\mathbf{u} = u_r \hat{\mathbf{e}}_r + u_z \hat{\mathbf{e}}_z,
\]

evaluate the expressions needed in the vector definition of the strain tensor.

3.34 A special case of Exercise 3.33.

3.35 Begin with the requirement that

\[
\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial^2 u_i}{\partial x_k \partial x_j}.
\]

Thus, \(u_{i,jk}\) is symmetric in \(j\) and \(k\). Hence,

\[
0 = \frac{\partial^2 u_i}{\partial x_j \partial x_k} \varepsilon_{jkl} = \varepsilon_{jkl} \frac{\partial}{\partial x_k} \left( \frac{\partial u_i}{\partial x_j} \right) = \varepsilon_{jkl} \frac{\partial}{\partial x_k} (\varepsilon_{ij} + \omega_{ij}).
\]

Differentiate with respect to \(x_m\) and multiply with \(\varepsilon_{imn}\) to obtain

\[
0 = \varepsilon_{jkl} \varepsilon_{imn} (\varepsilon_{ij,km} + \omega_{ij,km}) = \varepsilon_{jkl} \varepsilon_{imn} \varepsilon_{ij,km},
\]

which is the component form the required identity. By multiplying with \(\hat{\mathbf{e}}_n\) and \(\hat{\mathbf{e}}_l\), one can obtain the vector identity.

Chapter 4

4.1 \(V = (m_1 + m_2)g L_1 (1 - \cos \theta_1) + m_2 g L_2 (1 - \cos \theta_2)\).

4.2 \(V = (WL/4)(3\theta_1^2 + 5\theta_2^2 + \theta_3^2) + \frac{1}{2} \left[ k_1 \theta_1^2 + k_2 (\theta_2 - \theta_1)^2 + k_3 (\theta_3 - \theta_2)^2 \right].\)

4.3 The increment of complementary work done is given by \(\Delta W^* = (u + v \theta) \times \Delta F_x + [v - (L + u \theta)] \Delta F_y\) (assuming small \(\theta\)).
5.10  \( \theta_0 = -(M_0 L^4 / 4EI) + (q_0 L^3 / 48EI). \)

5.11

\[
w_c - w_e = -\left( \frac{25F_0 L^3 + q_0 L^4}{768EI} \right).
\]

Note that this is one of many relations between \( w_c \) and \( w_e \) that can be obtained depending on the choice of the virtual forces. It can be verified that the above relationship is exact.

5.12  \( w_B = (5F_0 a^3 / 6EI). \)

5.13 The force \( F_C \) in the cable is

\[
F_C = \left( \frac{L_c}{E_c A_c} + \frac{L_b}{E_b A_b} \cos^2 \alpha + \frac{L_b^3}{3E_b I_b} \sin^2 \alpha \right)^{-1} \left[ \frac{5L_b^3}{48E_b I_b} \sin \alpha \right] F_0.
\]

5.14 The slope at \( B \) is \( \theta_B = (170.67 / EI) \) clockwise.

5.15 The structure has bending and torsional deformations (see Example 5.14):

\[
w_A = \frac{1}{EI} \left[ \frac{Qa^3}{3} + \frac{(P + Q)b^3}{3} \right] + \frac{1}{GJ} (Qa^2b).
\]

5.16 The displacements are

\[
v = \frac{1}{6EI} \left[ P_v (2b^3 + 6ab^2) + 3P_h a^2b \right] + \frac{P_v a}{EA},
\]

\[
u = \frac{1}{6EI} \left( 3P_v a^2b + 2P_h a^3 \right) + \frac{P_h b}{EA}.
\]

5.17 We have \( u = 0 \) and \( v = (2595/32)(P^2 / A^2 E^2) \), where \( L_1 = 60 \) in., \( L_3 = 48 \) in., and \( L_4 = 36 \) in. (with the understanding that \( A \) and \( E \) are in sq. in. and psi, respectively).

5.18 \( (P_C = 4M_0 / 11L). \)

5.19

\[
w(L) = \frac{q_0 L^4}{30EI} \left( \frac{1}{1 + (kL_3/3EI)} \right).
\]

As a special case, the deflection at the free end of the beam when not supported by a spring is obtained by setting \( k = 0. \)

5.20 The displacements are

\[
w = \frac{PR^3 \pi}{4EI}, \quad u = \frac{PR^3}{2EI}.
\]
5.33 The deflection at the midspan is
\[ w_0 \left( \frac{L}{2} \right) = - \left( \frac{5F_0 L^3}{48EI} + \frac{17g_0 L^4}{384EI} \right). \]

5.34 From Maxwell's theorem, we have
\[ F_B \cdot w_{BA} + F_C \cdot w_{CA} = F_A \cdot w_{AB} + F_A \cdot w_{AC} = F_A \cdot w_A, \]
\[ w_A = w_{AB} + w_{AC}, \quad F_A = 4,000 \text{ lb}, \quad F_B = 4,500 \text{ lb}, \quad F_C = 2,000 \text{ lb}. \]

Chapter 6

6.1 The Lagrangian \( L \) of the system is
\[ L = -U = -\frac{1}{2} \left[ k_1 x_1^2 + (k_2 + k_3)(x_2 - x_1)^2 + k_4(x_3 - x_2)^2 \right]. \]

6.3 The rate of the complementary Lagrangian function is
\[ \delta L^* = -\dot{x}_1 \delta F_k - \dot{x}_2 \delta F_d + \dot{x} \delta F = \left( -\frac{\dot{F}}{k_1} - \frac{F}{\eta} + x \right) \delta F. \]

6.4 The Euler–Lagrange equations are given by
\[ \delta \theta_1: \quad -m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g L_1 \sin \theta_1 \]
\[ -\frac{d}{dt} \left[ (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 L_1 L_2 \cos(\theta_1 - \theta_2) \dot{\theta}_2 \right] = 0, \]
\[ \delta \theta_2: \quad m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g L_2 \sin \theta_2 \]
\[ -m_2 \frac{d}{dt} \left[ l_2^2 \dot{\theta}_2 + L_1 L_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \right] = 0. \]

6.5 The total kinetic energy of the rigid bar assemblage is
\[ K = \frac{W L^2}{6g} \left( 2 \dot{\theta}_1^2 + 4 \dot{\theta}_2^2 + \dot{\theta}_3^2 + 9 \dot{\theta}_1 \dot{\theta}_2 + 3 \dot{\theta}_2 \dot{\theta}_3 + 3 \dot{\theta}_1 \dot{\theta}_3 \right). \]

6.6 The Lagrangian function is
\[ L = \frac{1}{2} m_1 \left[ l_1^2 \dot{\theta}_1^2 + x^2 - 2x l_1 \dot{\theta}_1 \sin \theta \right] + \frac{1}{2} m_2 \dot{x}^2 \]
\[ + m_1 g \left( x - l_1 \cos \theta \right) + m_2 g x + \frac{1}{2} k (x + h)^2, \]
where \( h \) is the elongation in the spring due to the masses:
\[ h = \frac{g}{k} (m_1 + m_2). \]
6.7 The Lagrangian is given by
\[ L = \frac{1}{2} m_1 \dot{u}_1^2 + \frac{1}{2} m_2 \left( \dot{u}_1^2 + \dot{u}_2^2 - \sqrt{2} \dot{u}_1 \dot{u}_2 \right) + \frac{1}{\sqrt{2}} m_2 g u_2. \]

6.9 The equation of motion is
\[ \ddot{x}_1 = \left( \frac{m_1 - m_2}{m_1 + m_2} \right) g. \]

6.10 The equation of motion is
\[ \ddot{x}_1 = \left( \frac{m_1 - m_2 - m_3}{m_1 + m_2 + m_3} \right) g. \]

6.11 \[ \ddot{x}_1 = g/23. \]

6.12 The equations of motion are
\[ m (\ddot{x} + l \dddot{\theta} \cos \theta - l \dddot{\theta}^2 \sin \theta) + kx = F, \]
\[ m [\ddot{x} \cos \theta + (l^2 + \Omega^2) \dddot{\theta}] + mgl \sin \theta = 2 \pi F \cos \theta. \]

6.13 The natural boundary conditions become
\[ \rho I \frac{\partial^3 w_0}{\partial x \partial t^2} + m \frac{\partial^2 w_0}{\partial t^2} - \frac{\partial}{\partial x} \left( E I \frac{\partial^2 w_0}{\partial x^2} \right) = 0, \quad \text{at } x = L, \]
\[ E I \frac{\partial^2 w_0}{\partial x^2} + J \frac{\partial^3 w_0}{\partial x \partial t^2} = 0, \quad \text{at } x = L. \]

6.14 The Euler–Lagrange equations are
\[ \delta u_0: \quad - \frac{\partial N_{xx}}{\partial x} - f + \frac{\partial}{\partial t} \left( m_0 \frac{\partial u_0}{\partial t} \right) = 0, \]
\[ \delta w_0: \quad - \frac{\partial Q_x}{\partial x} - q + \frac{\partial}{\partial t} \left( m_0 \frac{\partial w_0}{\partial t} \right) = 0, \]
\[ \delta \phi: \quad - \frac{\partial M_{xx}}{\partial x} + Q_x + \frac{\partial}{\partial t} \left( m_0 \frac{\partial \phi}{\partial t} \right) = 0. \]

The boundary expressions indicate that \( u_0, w_0, \) and \( \phi \) are the primary variables and \( N_{xx}, M_{xx}, \) and \( Q_x \) are the secondary variables of the problem. Thus the natural boundary conditions involve specifying \( M_{xx} \) and \( Q_x \).

6.15 The stress resultants are
\[ M_{xx} = E I \frac{\partial \phi}{\partial x}, \quad Q_x = K_A \left( \frac{\partial w_0}{\partial x} + \phi \right). \]
7.59 The eigenvalues are $\lambda_1 = 4.212$ and $\lambda_2 = 34.188$.

7.60 See Exercise 7.54.

7.61 Let $W_3(x) = c_1 x^2 + c_2 x^3 + c_3 x^4$, and obtain $c_3 = 0$ and
\[ c_1 = \frac{f_0 L^2}{4a_0} \quad \text{and} \quad c_2 = -\frac{f_0 L}{36a_0}. \]

7.62 The solution is
\[ U_1 = (-2\sqrt{2} + \sqrt{6})(1 - x^2) + \sqrt{2} = 1.0355 + 0.3789x^2. \]

7.63 See Exercises 7.53 and 7.59. The eigenvalues are $\lambda_1 = 3.785$ and $\lambda_2 = 19.753$.

7.64 The one-parameter approximation $U_1 = c_{11} \sin \pi x \sin \pi y$ yields $c_{11} = (f_0/2\pi^2)$, which is about half the value of the first term in Exercise 7.54 ($c_{11} = 8f_0/\pi^4$).

7.65 The solution is $U_1(x, y) = e^{-\sqrt{10}y} (x - x^2)$.

7.67 The solution is given by
\[ W_1(x, t) = c_1(0) \cos \alpha t \left(1 - \cos 2\pi x \right). \]

7.68 For solution of the form $U_1 = c_1(x)(y^2 - b^2)$, one obtains
\[ \lambda_1 = \frac{\pi^2}{(2a)^2} + \frac{10}{(2b)^2}, \]

whereas the exact value is
\[ \lambda = \frac{\pi^2}{(2a)^2} + \frac{\pi^2}{(2b)^2}. \]

7.69 For the two-parameter approximation of the form
\[ U_2 = [c_1(x) + c_2(x)y^2](y^2 - b^2), \]

one obtains
\[ \lambda_1 = \frac{\pi^2}{(2a)^2} + \frac{9.871}{(2b)^2}. \]

7.70 A two-parameter approximation of the form (which is more complete than the one-parameter approximation, although it has no effect on the derivative of the solution)
\[ \Psi_2 = c_1 + c_2(x^4 - 6x^2y^2 + y^4). \]
9.3 The solution is

\[ U_2 = 0.2 \times 10^{-3} \text{ in.}, \quad U_3 = -0.3333 \times 10^{-3} \text{ in.}, \quad U_4 = -0.8667 \times 10^{-3} \text{ in.} \]

The forces in each member can be computed from the element equations:

\[ P_2^{(1)} = 3,000 \text{ lb}, \quad P_2^{(2)} = -2,000 \text{ lb}, \quad P_2^{(3)} = -2,000 \text{ lb}. \]

9.4 The compression is \( U_2 = -0.002996 \approx 0.003 \) in. The element forces \( P_i^{(b)} \) and \( P_i^{(s)} \) are \( P_2^{(b)} = -551.59 \) lb and \( P_2^{(s)} = -778.41 \) lb. The stresses in steel and brass are \( \sigma_s = 22.47 \) ksi (compressive), \( \sigma_b = 11.24 \) ksi (compressive).

9.5 The displacements are \( U_2 = 0.44444 \) in. and \( U_3 = 1.94444 \) in.

9.6 The aluminum member is elongated and the steel member is compressed by the amount \( U_2 = 0.0134 \) in. The reaction forces and stresses are

\[ P_2^{(1)} = -21,052.6 \text{ lb}, \quad \sigma_1 = 6701.25 \text{ psi}, \]
\[ P_2^{(2)} = -78,947 \text{ lb}, \quad \sigma_2 = -44,675.1 \text{ psi}. \]

9.7 The displacement is \( U_2 = 0.3 \) mm. The forces and stresses in steel and aluminum pipes are

\[ P_2^{(1)} = 36.364 \text{ kN}, \quad \sigma_s = 606.06 \text{ MPa}, \]
\[ P_2^{(2)} = -63.636 \text{ kN}, \quad \sigma_a = -106.06 \text{ MPa}. \]

9.8 First we must find the force acting at point B to be \( P = 5000 \) lb downward. The solution is

\[ U_2 = u_B = 0.01167 \text{ in.}, \quad U_3 = u_C = 0.01967 \text{ in.} \]

9.9 The solution is

\[ U_2 = u_B = 0.01163 \text{ in.}, \quad U_3 = u_C = 0.01956 \text{ in.} \]

Note that if we had taken \( k = 10^5 \) lb/in., we would have obtained

\[ U_2 = u_B = 0.00957 \text{ in.}, \quad U_3 = u_C = 0.01255 \text{ in.} \]

Thus one may use the value of \( k \) to simulate elastic restraint at point C.

9.10 Exploit the symmetry about the middle of the beam and use two beam elements to analyze the problem. The generalized displacements are

\[ U_2 = -0.00312, \quad U_3 = 0.01077 \text{ in.}, \quad U_4 = -0.001833, \quad U_5 = 0.01673 \text{ in.} \]
The reaction force at the left support and the internal bending moment at the center are

\[ Q_1^{(1)} = -1200 \text{ lb (up), } Q_4^{(2)} = 8400 \text{ lb-in. (counterclockwise).} \]

9.11 The solution is

\[ U_3 = 0.03329 \text{ cm, } U_4 = 0.01438, \quad U_6 = 0.00376. \]

9.12 The solution is

\[ U_2 = -0.001792, \quad U_4 = 0.000512, \quad U_5 = 0.03686 \text{ in., } U_6 = -0.001408. \]

9.13 One-element mesh can be used. The solution is given by

\[ U_3 = \frac{q_0 L^4}{8EI \left(1 + \frac{kL^3}{3EI}\right)}, \quad U_4 = -\frac{q_0 L^3}{6EI \left(1 + \frac{kL^3}{3EI}\right)}. \]

9.14 This problem can be modeled with four elements with \( h_1 = h_2 = h_3 = h_4 = 5' \). The main objective here is to represent the applied loads appropriately. The global node 2 will have a downward load of 1,000 lb and bending moment of 1,000 ft-lb (CCW). The total size of the assembled global stiffness matrix is 10 \( \times \) 10. The solution is given by

\[ \tilde{U}_2 = 0.12187, \quad \tilde{U}_3 = -0.45660 \text{ ft, } \tilde{U}_4 = 0.03021, \quad \tilde{U}_5 = -0.32986 \text{ ft, } \tilde{U}_6 = -0.06979, \quad \tilde{U}_8 = 0.001042, \quad \tilde{U}_9 = -0.39583 \text{ ft, } \tilde{U}_{10} = 0.10521, \]

where \( \tilde{U}_i = U_i (EI \times 10^{-5}) \). The bending moment at \( x = 2.5 \) ft, for example, is given by

\[ M^c = EI \left. \frac{d^2 w}{dx^2} \right|_{x=2.5} = EI \sum_{i=1}^{4} u_i^1 \left. \frac{d^2 \phi_i^1}{dx^2} \right|_{x=2.5h}, \]

\[ = EI \left( U_2 \left. \frac{d^2 \phi_2^1}{dx^2} \right|_{x=2.5} + U_3 \left. \frac{d^2 \phi_3^1}{dx^2} \right|_{x=2.5} + U_4 \left. \frac{d^2 \phi_4^1}{dx^2} \right|_{x=2.5} \right) = 1833.33 \text{ lb-ft.} \]

The shear force at \( x = 2.5 \) ft is given by \( V^c = 733.33 \) lb.

9.15 The primary objective of this problem is to compute the force vector for element 1. One obtains \( (q_0 = 500 \text{ and } h = 5) \)

\[ \{ q^{(1)} \} = \frac{q_0 h}{60} \begin{bmatrix} 9 \\ -2h \\ 21 \\ 3h \end{bmatrix} = \begin{bmatrix} 375.00 \\ -416.67 \\ 875.00 \\ 625.00 \end{bmatrix}. \]