Variational Methods in Theoretical Mechanics

Second Edition

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Preface to the First Edition

This is a textbook written for use in a graduate-level course for students of mechanics and engineering science. It is designed to cover the essential features of modern variational methods and to demonstrate how a number of basic mathematical concepts can be used to produce a unified theory of variational mechanics. As prerequisite to using this text, we assume that the student is equipped with an introductory course in functional analysis at a level roughly equal to that covered, for example, in Kolmogorov and Fomin (Functional Analysis, Vol. I, Graylock, Rochester, 1957) and possibly a graduate-level course in continuum mechanics. Numerous references to supplementary material are listed throughout the book.

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